

THE CAUCHY PROBLEM FOR THE SYSTEM OF THE THERMOELASTICITY IN E^n

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ABSTRACT. In this paper, we considered the problem of analytical continuation of the solution of the system equations of the thermoelasticity in spacious bounded domain from its values and values of its strains on part of the boundary of this domain, i.e., the Cauchy's problem.

Key words: the Cauchy problem, system theory of elasticity, elliptic system, ill-posed problem, Carleman matrix, regularization.

1. INTRODUCTION

In this paper, we considered the problem of analytical continuation of the solution of the system equations of the thermoelasticity in spacious bounded domain from its values and values of its strains on part of the boundary of this domain, i.e., the Cauchy's problem.

Since, in many actual problems, either a part of the boundary is inaccessible for measurement of displacement and tensions or only some integral characteristic are available. In experimental study of the stress-strain state of actual constructions, we can make measurements only on the accessible part of the surface.

In a practical investigation of experimental dates or diagnostic moving object arise problems of estimation concerning deformed position of the object. Solution of the problems by using well known classical propositions is connected to difficulties of absence of experimental dates which is necessary for formulation of boundary value (classical) conditions.

Therefore it is necessary consider the problem of continuation for solution of elasticity system of equations to the domain by values of solutions and normal derivatives in the part of boundary of domain.

System equation of the thermoelasticity is elliptic. Therefore the problem Cauchy for this system is ill-posed. For ill-posed problems, one does not prove the existence theorem: the existence is assumed a priori. Moreover, the solution is assumed to belong to some given subset of the function space, usually a compact one [1]. The uniqueness of the solution follows from the general Holmgren theorem [2]. On establishing uniqueness in the article studio of ill-posed problems, one comes across important questions concerning the derivation of estimates of conditional stability and the construction of regularizing operators.

Our aim is to construct an approximate solution using the Carleman function method.

Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be points of the n -dimensional Euclidean space E^n , D a bounded simply connected domain in E^n , with piecewise-smooth boundary consisting of a piece Σ of the plane $y_n = 0$ and a smooth surface S lying in the half-space $y_n > 0$.

Suppose that $n + 1$ -component vector function

$U(x) = (u_1(x), \dots, u_n(x), u_{n+1}(x))$ satisfied in D the system equations of the thermoelasticity [3]:

$$(1.1) \quad B(\partial_x, \omega)U(x) = 0.$$

Where

$$B(\partial_x, \omega) = \|B_{kj}(\partial_x, \omega)\|_{(n+1) \times (n+1)},$$

moreover

$$B_{kj}(\partial_x, \omega) = \delta_{kj}(\mu\Delta + \rho\omega^2) + (\lambda + \mu)\frac{\partial^2}{\partial x_k \partial x_j}, \quad k, j = 1, \dots, n,$$

$$B_{k(n+1)}(\partial_x, \omega) = -\gamma\frac{\partial}{\partial x_{(n+1)}}, \quad k = 1, \dots, n,$$

$$B_{(n+1)j}(\partial_x, \omega) = i\omega\eta\frac{\partial}{\partial x_j}, \quad j = 1, \dots, n,$$

$$B_{(n+1)(n+1)}(\partial_x, \omega) = \Delta + \frac{i\omega}{\theta},$$

δ_{ij} is the Kronecker delta, and where $\lambda, \mu, \rho, \omega, \theta$ is coefficients which characterizing medium, satisfying the conditions

$$\mu > 0, \quad 3\lambda + 2\mu > 0, \quad \rho > 0, \quad \theta > 0, \quad \frac{\gamma}{\eta} > 0.$$

The system (1.1) may be write in the following way:

$$\begin{cases} \mu\Delta u + (\lambda + \mu)\text{graddiv}u - \gamma\text{grad}v + \rho\omega^2 u = 0, \\ \Delta v + \frac{i\omega}{\theta}v + i\omega\eta\text{div}u = 0, \end{cases}$$

where $U(x) = (u(x), v(x))$.

That system is elliptic. As, it characteristic matrix is

$$\chi(\xi) = \left\| \begin{array}{cccccc} (\lambda + \mu)\xi_1^2 + \mu \sum_{i=1}^n \xi_i^2 & (\alpha + \mu)\xi_1\xi_2 & \cdots & (\alpha + \mu)\xi_1\xi_n & 0 \\ (\alpha + \mu)\xi_2\xi_1 & (\lambda + \mu)\xi_2^2 + \mu \sum_{i=1}^n \xi_i^2 & \cdots & (\alpha + \mu)\xi_2\xi_n & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 \end{array} \right\|,$$

and for arbitrary real $\xi = (\xi_1, \dots, \xi_n)$ satisfying conditions $\sum_{i=1}^n \xi_i^2 = 1$, we have

$$\det \chi(\xi) = \mu^2(\lambda + \mu) \neq 0$$

Statement of the problem. Find a regular solution U of system (1.1) in the domain D using its Cauchy data on the surface S :

$$(1.2) \quad U(y) = f(y), \quad R(\partial_y, \nu(y))U(y) = g(y), \quad y \in S,$$

where $R(\partial_y, \nu(y))$ is the stress operator, i.e.,

$$R(\partial_y, \nu(y)) = \|R_{kj}(\partial_y, \nu(y), \gamma)\|_{(n+1) \times (n+1)} = \left\| \begin{array}{ccc} T & -\gamma\nu_1 & \\ & -\gamma\nu_2 & \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & -\gamma\nu_n \\ 0 & 0 & \frac{\partial}{\partial \nu} \end{array} \right\|,$$

$$T = T(\partial_y, \nu) = \|T_{kj}(\partial_y, \nu)\|_{n \times n},$$

$$T_{kj}(\partial_y, \nu) = \lambda \nu_k \frac{\partial}{\partial y_j} + \mu \nu_j(y) \frac{\partial}{\partial y_k} + (\mu + \lambda) \delta_{kj} \frac{\partial}{\partial \nu(y)}, \quad k, j = 1, \dots, n,$$

$\nu(y) = (\nu_1(y), \dots, \nu_n(y))$ is the unit outward normal vector on ∂D at a point y , $f = (f_1, \dots, f_{n+1})$, $g = (g_1, \dots, g_{n+1})$ are given continuous vector functions on S .

2. CONSTRUCTION OF THE MATRIX CARLEMAN AND APPROXIMATE SOLUTION FOR THE DOMAIN TYPE'S CAP

It is well known, that any regular solution $U(x)$ system (1.1) is specified by the formula

$$(2.1) \quad 2U(x) = \int_{\partial D} \left(\Psi(x-y, \omega) \{R(\partial_y, \nu(y))U(y)\} - \{ \tilde{R}(\partial_y, \nu(y)) \tilde{\Psi}(y-x, \omega) \}^* U(y) \right) ds_y, \quad x \in D,$$

where symbol $*$ – is denote of operation transposition, $\Psi(y, x)$ matrix of fundamental solutions system equation of steady-state oscillations of the thermoelasticity: where

$$\Psi(x, \omega) = \|\Psi_{kj}(x, \omega)\|_{(n+1) \times (n+1)},$$

$$\Psi_{kj}(x, \omega) = \sum_{l=1}^3 \left[(1 - \delta_{k(n+1)})(1 - \delta_{j(n+1)}) \left(\frac{\delta_{kl}}{2\pi\mu} \delta_{nl} - \alpha_l \frac{\partial^2}{\partial x_k \partial x_j} \right) + \beta_l \left(i\omega\eta\delta_{k(n+1)}(1 - \delta_{j(n+1)}) \frac{\partial}{\partial x_j} - \gamma\delta_{j(n+1)}(1 - \delta_{k(n+1)}) \frac{\partial}{\partial x_k} \right) + \delta_{k(n+1)}\delta_{j(n+1)}\gamma_l \right] \frac{\exp(i\lambda_l |x|)}{|x|},$$

$$\alpha_l = \frac{(-1)^l(1 - i\omega\theta^{-1}\lambda^{-1}(\delta_{1l} + \delta_{2l}))}{2\pi(\lambda + 2\mu)(\lambda_2^2 - \lambda_1^2)} - \frac{\delta_{3l}}{2\pi\rho\omega^2}, \quad l = 1, 2, 3; \quad \sum_{l=1}^3 \alpha_l = 0,$$

$$\beta_l = \frac{(-1)^l(\delta_{1l} + \delta_{2l})}{2\pi(\lambda + 2\mu)(\lambda_2^2 - \lambda_1^2)}, \quad l = 1, 2, 3; \quad \sum_{l=1}^3 \beta_l = 0,$$

$$\gamma_l = \frac{(-1)^l(\lambda_l^2 - k_1^2)(\delta_{1l} + \delta_{2l})}{2\pi(\lambda_2^2 - \lambda_1^2)}, \quad l = 1, 2, 3; \quad \sum_{l=1}^3 \gamma_l = 0, \quad k_1^2 = \rho\omega^2(\lambda + 2\mu)^{-1},$$

$$\tilde{\Psi}(x, \omega) = \|\tilde{\Psi}_{kj}(x, \omega)\|_{(n+1) \times (n+1)}, \quad \tilde{\Psi}_{kj}(x, \omega) = \Psi_{kj}(-x, \omega),$$

$$\tilde{R}(\partial_y, \nu(y)) = \left\| \begin{array}{ccc} T & -i\omega\nu_1 & \\ & -i\omega\nu_2 & \\ \dots & \dots & \dots \\ \dots & \dots & -i\omega\nu_n \\ 0 & 0 & \frac{\partial}{\partial \nu} \end{array} \right\|.$$

Definition. By the Carleman matrix of problem (1.1), (1.2) we mean an $(n+1) \times (n+1)$ matrix $\Pi(y, x, \omega, \tau)$ depending on the two points y, x and positive numerical number parameter τ satisfying the following two conditions:

$$1) \quad \Pi(y, x, \omega, \tau) = \Psi(x-y, \omega) + G(y, x, \tau),$$

where matrix $G(y, x, \tau)$ satisfies system (1.1) with respect to the variable y in the domain D , and $\Psi(y, x)$ is a matrix of the fundamental solutions of system (1.1);

$$2) \quad \int_{\partial D \setminus S} (|\Pi(y, x, \omega, \tau)| + |R(\partial_y, \nu)\Pi(y, x, \omega, \tau)|) ds_y \leq \varepsilon(\tau),$$

where $\varepsilon(\tau) \rightarrow 0$, as $\tau \rightarrow \infty$; here $|\Pi|$ is the Euclidean norm of the matrix $\Pi = \|\Pi_{ij}\|_{(n+1) \times (n+1)}$, i.e., $|\Pi| = (\sum_{i,j=1}^{n+1} \Pi_{ij}^2)^{\frac{1}{2}}$. In particular, $|U| = (\sum_{m=1}^{n+1} u_m^2)^{\frac{1}{2}}$.

From the definition Carleman matrix it follows that

Theorem 1. Any regular solution $U(x)$ of system (1.1) in the domain D is specified by the formula

$$(2.2) \quad 2U(x) = \int_{\partial D} (\Pi(y, x, \omega, \tau) \{R(\partial_y, \nu)U(y)\} - \{\tilde{R}(\partial_y, \nu)\Pi(y, x, \omega, \tau)\}^* U(y)) ds_y, \quad x \in D,$$

where $\Pi(y, x, \omega, \tau)$ is matrix Carleman.

Using the matrix Carleman, easily conclude the estimate stability of solution of the problem (1.1), (1.2) and also indicate effective method decision this problem as [6-8].

With a view to construct approximate solution of the problem (1.1), (1.2) we construct the following matrix:

$$(2.3) \quad \Pi(y, x, \omega) = \|\Pi_{kj}(y, x, \omega)\|_{(n+1) \times (n+1)},$$

$$\Pi_{kj}(y, x, \omega) = \sum_{l=1}^3 \left[(1 - \delta_{k(n+1)})(1 - \delta_{j(n+1)}) \left(\frac{\delta_{kj}}{2\pi\mu} \delta_{nl} - \alpha_l \frac{\partial^2}{\partial x_k \partial x_j} \right) + \beta_l \left(i\omega \eta \delta_{k(n+1)}(1 - \delta_{j(n+1)}) \frac{\partial}{\partial x_j} - \gamma \delta_{j(n+1)}(1 - \delta_{k(n+1)}) \frac{\partial}{\partial x_k} \right) + \delta_{k(n+1)} \delta_{j(n+1)} \gamma_l \right] \Phi(y, x, k_l)$$

where

$$(2.4) \quad C_n K(x_n) \Phi(y, x, k) = \int_0^\infty \text{Im} \left[\frac{K(i\sqrt{u^2 + s} + y_n)}{i\sqrt{u^2 + s} + y_n - x_n} \right] \frac{\psi(ku) du}{\sqrt{u^2 + s}},$$

$$\psi(ku) = \begin{cases} u J_0(ku), & n = 2m, \quad m \geq 1, \\ \cos ku, & n = 2m + 1, \quad m \geq 1, \end{cases} \quad J_0(u) \text{-Bessel function of order zero,}$$

$$s = (y_1 - x_1)^2 + \dots + (y_{n-1} - x_{n-1})^2, \quad C_2 = 2\pi$$

$$C_n = \begin{cases} (-1)^m \cdot 2^{-n} (n-2) \pi \omega_n (m-2)!, & n = 2m \\ (-1)^m \cdot 2^{-n} (n-2) \pi \omega_n (m-1)!, & n = 2m + 1. \end{cases}$$

$K(\omega)$, $\omega = u + iv$ (u, v are real), is an entire function taking real values on the real axis and satisfying the conditions $K(u) \neq \infty$, $|u| < \infty$, $K(u) \neq 0$, $\sup_{v \geq 1} |\exp \nu| \text{Im} K^{(p)}(\omega)| = M(p, u) < \infty$, $p = 0, \dots, m$, $u \in R^1$.

In work [4] proved.

Lemma 1. For function $\Phi(y, x, k)$ the formula is valid

$$C_n \Phi(y, x, k) = \varphi_n(ikr) + g_n(y, x, k), \quad r = |y - x|,$$

where φ_n -fundamental solution Helmholtz equation, $g_n(y, x, k)$ is a regular function that is defined for all y and x satisfies Helmholtz equation: $\Delta(\partial_y)g_n - k^2 g_n = 0$.

In (2.4) we assume the function $K(\omega) = \exp(\tau\omega)$. Then

$$\Phi(y, x, k) = \Phi_\tau(y - x, k),$$

$$\begin{aligned}
(2.5) \quad C_n \Phi_\tau(y-x, k) &= \frac{\partial^{m-1}}{\partial s^{m-1}} \int_0^\infty \operatorname{Im} \left[\frac{\exp \tau(i\sqrt{u^2+s} + y_n - x_n)}{i\sqrt{u^2+s} + y_n - x_n} \right] \frac{\psi(ku) du}{\sqrt{u^2+s}} = \\
&= \exp \tau(y_n - x_n) \frac{\partial^{m-1}}{\partial s^{m-1}} \int_0^\infty \left[-\cos \tau \sqrt{u^2 + \alpha^2} + (y_n - x_n) \frac{\sin \tau \sqrt{u^2 + s}}{\sqrt{u^2 + s}} \right] \psi(ku) du, \\
\Phi'_\tau(y-x, k) &= \frac{\partial \Phi_\tau}{\partial \tau}.
\end{aligned}$$

$$C_n \Phi'_\tau(y-x, k) = \exp \tau(y_n - x_n) \frac{\partial^{m-1}}{\partial s^{m-1}} \int_0^\infty \frac{\sin \tau \sqrt{u^2 + s}}{\sqrt{u^2 + s}} \psi(ku) du,$$

$$C_n \Phi'_\tau(y-x, k) = \exp \tau(y_n - x_n) \frac{\partial^{m-1}}{\partial s^{m-1}} \psi'_\tau(k, s),$$

$$\psi'_\tau = \begin{cases} 0, & \tau < k \\ \cos \sqrt{s(\tau^2 - k^2)}, & n = 2m \\ \frac{1}{2} \pi J_0(\sqrt{s(\tau^2 - k^2)}), & \tau > k \end{cases}$$

Now in formul (2.3), and (2.4) to take $\Phi(y, x, k) = \Phi_\tau(y-x, k)$, we construct matrix $\Pi(y, x, \omega) = \Pi(y, x, \omega, \tau)$

From Lemma 1 we obtain.

Lemma 2. *The matrix $\Pi(y, x, \omega, \tau)$ given by (2.3) and (2.4) is Carleman's matrix for problem (1.1), (1.2).*

Proof. By (2.3), (2.4) and Lemma 1 we have

$$\Pi(y, x, \omega, \tau) = \Psi(y, x, \omega) + G(y, x, \tau),$$

where

$$G(y, x, \tau) = \|G_{kj}(y, x, \tau)\|,$$

$$\begin{aligned}
G_{kj}(y, x, \tau) &= \sum_{l=1}^3 \left[(1 - \delta_{k(n+1)})(1 - \delta_{j(n+1)}) \left(\frac{\delta_{kj}}{2\pi\mu} \delta_{nl} - \alpha_l \frac{\partial^2}{\partial x_k \partial x_j} \right) + \right. \\
&\quad + \beta_l \left(i\omega \eta \delta_{k(n+1)}(1 - \delta_{j(n+1)}) \frac{\partial}{\partial x_j} - \gamma \delta_{j(n+1)}(1 - \delta_{k(n+1)}) \frac{\partial}{\partial x_k} \right) + \\
&\quad \left. + \delta_{k(n+1)} \delta_{j(n+1)} \gamma_l \right] g_n(y, x, k_l, \tau), \quad k, j = 1, \dots, n+1
\end{aligned}$$

By a straightforward calculation, we can verify that the matrix $G(y, x, \tau)$ satisfies system (1.1) with respect to the variable y everywhere in D . By using (2.3), (2.4) and (2.5) we obtain

$$(2.6) \quad \int_{\partial D \setminus S} \left(\left| \Pi(y, x, \omega, \tau) \right| + \left| R(\partial_y, \nu) \Pi(y, x, \omega, \tau) \right| \right) ds_y \leq C_1(x) \tau^m \exp(-\tau x_n),$$

where $C_1(x)$ some bounded function inside D . The lemma is thereby proved.

Let us set

$$(2.7) \quad 2U_\tau(x) = \int_S [\Pi(y, x, \omega, \tau) \{R(\partial_y, \nu)U(y)\} - \{\tilde{R}(\partial_y, n)\Pi(y, x, \omega, \tau)\}^* U(y)] ds_y.$$

The following theorem holds.

Theorem 1. *Let $U(x)$ be a regular solution of system (1.1) in D such that*

$$(2.8) \quad |U(y)| + |R(\partial_y, \nu)U(y)| \leq M, \quad y \in \partial D \setminus S.$$

Then for $\tau \geq 1$ the following estimate is valid:

$$|U(y) - U_\tau(y)| \leq MC_2(x)\tau^m \exp(-\tau x_n).$$

Proof. By formula (2.2) and (2.7), we have

$$2|U(x) - U_\tau(x)| = \int_{\partial D \setminus S} [\Pi(y, x, \omega, \tau)\{R(\partial_y, \nu)U(y)\} - \{\tilde{R}(\partial_y, \nu)\Pi(y, x, \omega, \tau)\}^* U(y)] ds_y.$$

Now on the basis of (2.6) and (2.8) we obtain required inequality. The theorem is thereby proved.

Now we write out a result that allows us to calculate $U(x)$ approximately if, instead of $U(y)$ and $R(\partial_y, \nu)U(y)$, their continuous approximations $f_\delta(y)$ and $g_\delta(y)$ are given on the surface S :

$$(2.9) \quad \max_S |f(y) - f_\delta(y)| + \max_S |R(\partial_y, \nu)U(y) - g_\delta(y)| \leq \delta, \quad 0 < \delta < 1.$$

We define a function $U_{\tau\delta}(x)$ by setting

$$(2.10) \quad 2U_{\tau\delta}(x) = \int_S [\Pi(y, x, \omega, \tau)g_\delta(y) - \{\tilde{R}(\partial_y, \nu)\Pi(y, x, \omega, \tau)\}^* f_\delta(y)] ds_y,$$

where

$$\tau = \frac{1}{x_n^0} \ln \frac{M}{\delta}, \quad x_n^0 = \max_D x_n, \quad x_n > 0.$$

Theorem 2. Let $U(x)$ be a regular solution of system (1.1) in D satisfying condition (2.8). Then the following estimate is valid:

$$|U(x) - U_{\tau\delta}(x)| \leq C_3(x) \delta^{\frac{x_n}{x_n^0}} \left(\ln \frac{M}{\delta} \right)^m, \quad x \in D.$$

Proof. From formula (2.2) and (2.10) we have

$$\begin{aligned} 2(U(x) - U_{\tau\delta}(x)) &= \int_S [\Pi(y, x, \omega, \tau)\{R(\partial_y, \nu)U(y) - g_\delta(y)\} - \\ &\quad - \{\tilde{R}(\partial_y, \nu)\Pi(y, x, \omega, \tau)\}^* (U(y) - f_\delta(y))] ds_y + \\ &\quad + \int_{\partial D \setminus S} [\Pi(y, x, \omega, \tau)\{R(\partial_y, \nu)U(y)\} - \{\tilde{R}(\partial_y, \nu)\Pi(y, x, \omega, \tau)\}^* U(y)] ds_y. \end{aligned}$$

By the assumption of the theorem and inequalities (2.6), (2.8) and (2.9) for the any $x \in D$, we obtain

$$\begin{aligned} |U(x) - U_{\tau\delta}(x)| &= C_2(x) \delta \tau^m \exp \tau(x_n^0 - x_n) + C_1(x) \tau^m \exp(-\tau x_n) \leq \\ &\leq C_3(x) \tau^m (M + \delta \exp \tau x_n^0) \exp(-\tau x_n). \end{aligned}$$

Now, it to take $\tau = \frac{1}{x_n^0} \ln \frac{M}{\delta}$, then we obtain to proof theorem. The theorem is thereby proved.

Theorem 3. Let $U(x)$ be a regular solution of system (1.1) in D satisfying conditions

$$\begin{aligned} |U(y)| + |R(\partial_y, \nu)U(y)| &\leq M, \quad y \in \partial D \setminus S, \\ |U(y)| + |R(\partial_y, \nu)U(y)| &\leq \delta, \quad 0 < \delta < 1, \quad y \in S. \end{aligned}$$

Then

$$|U(x)| \leq C_4(x) \delta^{\frac{x_n}{x_n^0}} \left(\ln\left(\frac{M}{\delta}\right) \right)^m,$$

where $C_4(x) = \tilde{C} \int_{\partial D} \frac{1}{r^n} ds_y$, \tilde{C} – constant depending on λ, μ, ω .

Proof. On the basis of Theorem 2 we obtain

$$|U(x)| \leq |U_\tau(x)| + MC_2(x) \tau^m \exp(-\tau x_n).$$

Next from the condition theorem and (2.3), (2.4) we obtain

$$\begin{aligned} |2U_\tau(x)| &= \left| \int_S [\Pi(y, x, \omega, \tau) \{R(\partial_y, \nu)U(y)\} - \{\tilde{R}(\partial_y, \nu)\Pi(y, x, \omega, \tau)\}^*(U(y))] ds_y \right| \leq \\ &\leq \int_S (|\Pi(y, x, \omega, \tau)| + |R(\partial_y, \nu)\Pi(y, x, \omega, \tau)|) (|U(y)| + |R(\partial_y, \nu)U(y)|) ds_y \leq \\ &\leq \delta \int_S (|\Pi(y, x, \omega, \tau)| + |R(\partial_y, \nu)\Pi(y, x, \omega, \tau)|) ds_y \leq C_3(x) \delta \tau^m \exp(\tau x_n^0 - \tau x_n). \end{aligned}$$

Then

$$|U(x)| \leq C_4(x) \tau^m \exp(-\tau x_n) (M + \delta \exp \tau x_n^0).$$

Next if we take $\tau = \frac{1}{x_n^0} \ln \frac{M}{\delta}$, then we obtain stability estimate:

$$|U(x)| \leq C_4(x) \delta^{\frac{x_n}{x_n^0}} \left(\ln\left(\frac{M}{\delta}\right) \right)^m.$$

The theorem is thereby proved.

From proved above theorems we obtain

Corollary 1. *The limit relation*

$$\lim_{\tau \rightarrow \infty} U_\tau(x) = U(x), \quad \lim_{\delta \rightarrow 0} U_\tau(x) = U(x)$$

hold uniformly on each compact subset of D .

3. REGULARIZATION OF SOLUTION OF THE PROBLEM (1.1), (1.2) FOR A DOMAIN OF CONE TYPE

Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be points in E^n , D_ρ be a bounded simply connected domain in E^n whose boundary consists of a cone surface

$$\Sigma: \quad \alpha_1 = \tau_\rho y_n, \quad \alpha_1^2 = y_1^2 + \dots + y_{n-1}^2, \quad \tau_\rho = tg \frac{\pi}{2\rho}, \quad y_n > 0, \quad \rho > 1$$

and a smooth surface S , lying in the cone. Assume $x_0 = (0, \dots, 0, x_n) \in D_\rho$.

We construct Karleman matrix. In formula (2.3), (2.4) to take

$$K(\omega) = E_\rho [\tau (\omega - x_n)], \quad \tau > 0, \quad \rho > 1.$$

Then

$$\Phi(y, x, k) = \Phi_\tau(y - x, k), \quad k > 0$$

$$(3.1) \quad C_n \Phi_\tau(y-x, k) = \frac{\partial^{m-1}}{\partial s^{m-1}} \int_0^\infty \operatorname{Im} \left[\frac{E_\rho(\tau(i\sqrt{u^2+s} + y_n - x_n))}{i\sqrt{u^2+s} + y_n - x_n} \right] \frac{\psi(ku) du}{\sqrt{u^2+s}}$$

$$\Phi'_\tau(y-x, k) = \frac{\partial \Phi_\tau}{\partial \tau}.$$

$$C_n \Phi'_\tau(y-x, k) = \frac{\partial^{m-1}}{\partial s^{m-1}} \int_0^\infty \operatorname{Im} \left\{ E'_\rho \left[\tau(i\sqrt{u^2+s} + y_n - x_n) \right] \right\} \frac{\psi(ku) du}{\sqrt{u^2+s}},$$

where $E_\rho(w)$ — Mittag-Löffer's entire function [5]. For the functions $\Phi_\tau(y-x, k)$ holds Lemma 1 and Lemma 2.

Now again to denote by $U_\tau(x)$, $U_{\tau\delta}(x)$ as (2.7) and (2.10). Then holds analogical theorem as **Theorem 1,2,3**.

For $n = 3$ we reduce entirely.

Suppose that D_ρ is bounded simple connected domain from E^3 with boundary consisting of part Σ of the surface of the cone

$$y_1^2 + y_2^2 = \tau_\rho y_3^2, \quad \tau_\rho = tg \frac{\pi}{2\rho}, \quad \rho > 1, \quad y_3 > 0,$$

and of a smooth portion of the surface S lying inside the cone. Assume $x_0 = (0, 0, x_3) \in D_\rho$.

We construct Carleman's matrix. In formula (2.3), (2.4) we take

$$(3.2) \quad \Phi_\tau(y, x, k) = \frac{1}{4\pi^2 E_\rho(\tau^{\frac{1}{\rho}} x_3)} \int_0^\infty \operatorname{Im} \frac{E_\rho(\tau^{\frac{1}{\rho}} w)}{i\sqrt{u^2+s} + y_3 - x_3} \frac{\cos ku du}{\sqrt{u^2+s}},$$

where $w = i\sqrt{u^2+s} + y_3$. For the functions $\Phi_\tau(y, x, k)$ holds Lemma 1.

It follows from the properties of $E_\rho(w)$ that for $y \in \Sigma$, $0 < u < \infty$ the function $\Phi_\tau(y, x, k)$ defined by (3.1) its gradient and second partial derivatives

$$\frac{\partial^2 \Phi_\tau(y, x, k)}{\partial y_k \partial y_j}, \quad k, j = 1, 2, 3,$$

tend to zero as $\tau \rightarrow \infty$ for a fixed $x \in D_\rho$.

Then from (2.3) we find that the matrix $\Pi(y, x, \omega, \tau)$ and its stresses $R(\partial_y, \nu)\Pi(y, x, \omega, \tau)$ also tend to zero as $\tau \rightarrow \infty$ on $y \in \Sigma$, i.e., $\Pi(y, x, \omega, \tau)$ — is the Carleman matrix for the domain D_ρ and the part Σ of the boundary.

For the $U(x)$ — regular solution system (1.1) following integral formula holds

$$2U(x) = \int_{\partial D_\rho} [\Pi(y, x, \omega, \tau) \{R(\partial_y, \nu)U(y)\} - \{\tilde{R}(\partial_y, \nu)\Pi(y, x, \omega, \tau)\}^* U(y)] ds_y.$$

By $x \in D_\rho$ we denote $U_\tau(x)$ follows:

$$(3.3) \quad 2U_\tau(x) = \int_S [\Pi(y, x, \omega, \tau) \{R(\partial_y, \nu)U(y)\} - \{\tilde{R}(\partial_y, \nu)\Pi(y, x, \omega, \tau)\}^* U(y)] ds_y.$$

The following theorem holds.

Theorem 4. Let $U(x)$ be a regular solution of system (1.1) in D_ρ such that

$$(3.4) \quad |U(y)| + |R(\partial_y, \nu)U(y)| \leq M, \quad y \in \Sigma.$$

Then for $\tau \geq 1$ the following estimate is valid:

$$|U(x_0) - U_\tau(x_0)| \leq MC_\rho(x_0)\tau^3 \exp(-\tau x_3^\rho),$$

where $x_0 = (0, 0, x_3) \in D_\rho$, $x_3 > 0$,

$$C_\rho(x_0) = C_\rho \int_{\Sigma} \frac{1}{r_0^3} ds_y, \quad r_0 = |y - x_0|, \quad C_\rho - \text{constant}.$$

Proof. By analogy with proved Theorem 2 and Theorem 3 from (3.3) and (3.4) we obtain

$$|U(x_0) - U_\tau(x_0)| \leq M \int_{\Sigma} [|\Pi(y, x_0, \omega, \tau)| + |\tilde{R}(\partial_y, \nu)\Pi(y, x_0, \omega, \tau)|] ds_y.$$

By formula (3.2) we have following inequality:

$$\begin{aligned} |\Phi_\tau(y, x, k)| &\leq C_\rho^{(1)} E_\rho^{-1} (\tau^{\frac{1}{\rho}} x_3) r^{-1}, \\ \left| \frac{\partial \Phi_\tau(y, x, k)}{\partial y_i} \right| &\leq C_\rho^{(2)} \tau E_\rho^{-1} (\tau^{\frac{1}{\rho}} x_3) r^{-2} \\ \left| \frac{\partial^2 \Phi_\tau(y, x, k)}{\partial y_k \partial y_j} \right| &\leq C_\rho^{(3)} \tau^2 E_\rho^{-1} (\tau^{\frac{1}{\rho}} x_3) r^{-3}. \end{aligned}$$

Then from (2.3)

$$|\Pi(y, x, \omega, \tau)| \leq C_\rho^{(4)} \tau^2 E_\rho^{-1} (\tau^{\frac{1}{\rho}} x_3) r^{-3},$$

$$|\tilde{R}(\partial_y, \nu)\Pi(y, x, \omega, \tau)| \leq C_\rho^{(5)} \tau^3 E_\rho^{-1} (\tau^{\frac{1}{\rho}} x_3) r^{-4}.$$

Therefore we obtain

$$|U(x_0) - U_\tau(x_0)| \leq MC_\rho(x_0)\tau^3 \exp(-\tau x_3^\rho),$$

where

$$C_\rho(x_0) = C_\rho \int_{\Sigma} \frac{1}{r_0^3} ds_y, \quad r_0 = |y - x_0|, \quad C_\rho - \text{constant}.$$

The theorem is thereby proved.

Suppose that instead of $U(y)$ and $R(\partial_y, \nu)U(y)$ gives their continuous approximations $f_\delta(y)$ and $g_\delta(y)$ such that

$$\max_S |U(y) - f_\delta(y)| + \max_S |T(\partial_y, \nu)U(y) - g_\delta(y)| \leq \delta, \quad 0 < \delta < 1.$$

Define the function $U_{\tau\delta}(x)$ by

$$2U_{\tau\delta}(x) = \int_S [\Pi(y, x, \omega, \tau)g_\delta(y) - \{\tilde{R}(\partial_y, \nu)\Pi(y, x, \omega, \tau)\}^* f_\delta(y)] ds_y,$$

The following theorem holds

Theorem 5. Let $U(x)$ is a regular solution of system (1.1) in the domain D_ρ satisfying the condition (3.4), then

$$|U(x_0) - U_{\tau\delta}(x_0)| \leq C_\rho(x_0)\delta^q \left(\ln \frac{M}{\delta}\right)^3,$$

where $\tau = (\tau_\rho R)^{-\rho} \ln \frac{M}{\delta}$, $R^\rho = \max_S \operatorname{Re}(i\sqrt{s} + y_3)^\rho$,

$$q = \left(\frac{x_3}{R}\right)^\rho, \quad C_\rho(x_0) = C_\rho \int_\Sigma \left[\frac{1}{r_0^3} + \frac{1}{r_0^4} \right] ds_y.$$

The proof theorem is similar to those of Theorem 3 and 4.

Corollary 2. *The limit relation*

$$\lim_{\tau \rightarrow \infty} U_\tau(x) = U(x), \quad \lim_{\delta \rightarrow 0} U_{\tau\delta}(x) = U(x)$$

hold uniformly on each compact subset of D_ρ .

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